Appendix D

LOCAL CONSERVATION BALANCES

Though numerical models of unsteady, gradually-varied flow are theoretically based on the continuous Equations 2.4.1 and 2.4.2, the equations actually solved by numerical model code are descretized transformations of these equations. Independent confirmation that the numerical predictions do indeed satisfy Equations 2.4.1 and 2.4.2 is a useful independent check on a numerical code.

A numerical model provides predictions of η and Q at discrete positions and times, $\eta_i^n = \eta(x_i, t_n)$ and $Q_i^n = Q(x_i, t_n)$. The x_i are the fixed locations of spatial nodes. They are frequently uniformly spaced, but this is not necessary. The times t_n also are generally uniformly spaced, though this is again unnecessary.

To evaluate the contributing terms in the mass and momentum balances, local estimates of time and space derivatives of the dependent variables are required. And these must be determined in a manner that is independent of the numerical code but, at the same time, does not compromise the numerical predictions. The truncation errors of common numerical codes typically do not exceed second order in Δx and Δt. This is equivalent to local interpolation following a Taylor series expansion, truncated after the second order terms. Such a truncated Taylor series expansion is a bi-quadratic polynomial in x and t.

Accordingly, the local behavior of the dependent variables is assumed to follow a bi-quadratic polynomial. For $\eta(x, t)$, this is

$$\eta(x, t) = \eta_{00} + \eta_{01}t + \eta_{10}x + \eta_{02}t^2 + \eta_{11}xt + \eta_{20}x^2 \tag{D.0.1}$$

at each local node, which is at $x = 0$ and $t = 0$ in the local coordinate system. The six polynomial coefficients, η_{00}, η_{01}, η_{10}, η_{02}, η_{11}, η_{20}, are determined from the nine known and neighboring nodal predictions for η, specifically η_i^n, η_i^{n-1}, η_i^{n+1}, η_{i-1}^{n-1}, η_{i-1}^n, η_{i-1}^{n+1}, η_{i+1}^{n-1}, η_{i+1}^n, and η_{i+1}^{n+1}. Using the five nearest nodes, η_i^n, η_i^{n-1}, η_i^{n+1}, η_{i-1}^n and η_{i+1}^n, in Equation D.0.1 defines five simultaneous linear algebraic equations and a direct solution for the polynomial coefficients, η_{00}, η_{01}, η_{10}, η_{02} and η_{20}.

The cross coefficient η_{11} is determined by least squares from Equation D.0.1 at the four more distant nodes, η_{i-1}^{n-1}, η_{i-1}^{n+1}, η_{i+1}^{n-1}, and η_{i+1}^{n+1}. However, it does not subsequently appear in the approximations to either of the conservation equations.

$Q(x, t)$ is similarly defined with polynomial coefficients $Q_{00} \ldots Q_{02}$. Dependent channel geometry parameters are similarly defined, $b(x, t)$ with coefficients $b_{00} \ldots b_{02}$, $A(x, t)$ with coefficients

$A_{00} \dots A_{02}$, and $P(x,t)$ with coefficients $P_{00} \dots P_{02}$.

With η, Q, b, A and P analytically defined in the neighborhood of the node, the terms in Equations 2.4.1 and 2.4.2 are evaluated by direct substitution. At the node, $x = 0$ and $t = 0$, the Mass Equation 2.4.1 becomes

$$b_{00}\eta_{01} + Q_{10} = 0 \tag{D.0.2}$$

which is Storage + Advection = 0. The sum, Σ = Storage + Advection, should be zero throughout.

Similarly, the Momentum Equation 2.4.2 becomes

$$Q_{01} + \left(2\frac{Q_{00}}{A_{00}}Q_{10} - \frac{Q_{00}^2}{A_{00}^2}A_{10}\right) = -gA_{00}\eta_{10} + \text{Friction} \tag{D.0.3}$$

which are Temporal Inertia + Advective Inertia = Gravity + Friction. The Friction term is

$$\text{Friction} = \begin{cases} -\dfrac{f}{8}\dfrac{|Q_{00}|Q_{00}}{A_{00}^2}P_{00} & \text{Darcy Weisbach} \\ -\dfrac{gn^2}{(A_{00}/P_{00})^{1/3}}\dfrac{|Q_{00}|Q_{00}}{A_{00}^2}P_{00} & \text{Manning [SI]} \end{cases} \tag{D.0.4}$$

The sum, Σ = Temporal Inertia + Advective Inertia - Gravity - Friction, should be zero throughout.